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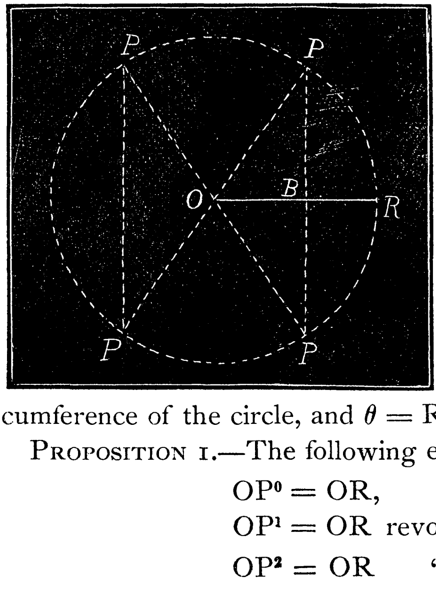
INVOLUTION AND EVOLUTION OF IMAGINARY QUANTITIES CONSIDERED GEOMETRICALLY.

BY PROF. W. D. HENKLE, SALEM, OHIO.

In the diagram let the radius of the circle be 1 and OR its initial position. Conceive the radius to revolve as the hands of a clock. Let OP represent any position of the radius.



Draw PB perpendicular to the line of initial direction. Put OB $=a$ and PB $=b$. Then $a+b\sqrt{-1}$ represents OP, a being positive when P is in the right half of the circumference and negative when P is in the left half of the circumference, and b being positive when P is in the upper half of the circumference and negative when P is in the lower half of the circumference. Let C represent the circumference of the circle, and $\theta =$ RP.

Proposition 1.—The following equations indicate the powers of OP:

$$\begin{aligned}
&\mathrm{OP}^0 = \mathrm{OR}, \\
&\mathrm{OP}^1 = \mathrm{OR} \text{ revolved through } 1\theta, \\
&\mathrm{OP}^2 = \mathrm{OR} \quad \text{“} \quad \text{“} \quad 2\theta, \\
&\mathrm{OP}^3 = \mathrm{OR} \quad \text{“} \quad \text{“} \quad 3\theta, \\
\text{and in general} \quad &\mathrm{OP}^n = \mathrm{OR} \quad \text{“} \quad \text{“} \quad n\theta.
\end{aligned}$$

Demonstration.—Since $a = \cos\theta$ and $b = \sin\theta$ we have

$$a + b\sqrt{-1} = \cos\theta + \sqrt{-1}\,\sin.\theta.$$

But we have by De Moivre's formula

$$(\cos\theta + \sqrt{-1}\,\sin\theta)^n = \cos n\theta + \sqrt{-1}\,\sin.n\theta.$$

Hence $\qquad \mathrm{OP}^n = \cos n\theta + \sqrt{-1}\,\sin.n\theta,$

or $\qquad \mathrm{OP}^n = \mathrm{O}R$ revolved through $n\theta$.

Proposition 2.—The following equations indicate the roots of OP:

$\mathrm{OP}^{\frac{1}{1}} = \mathrm{OR}$ revolved through $\frac{1}{1}\theta$,

$\mathrm{OP}^{\frac{1}{2}} = \mathrm{OR}$ “ “ $\frac{1}{2}\theta$,

$\mathrm{OP}^{\frac{1}{3}} = \mathrm{OR}$ “ “ $\frac{1}{3}\theta$,

and in general $\mathrm{OP}^{\frac{1}{n}} = \mathrm{OR}$ “ “ $\frac{1}{n}\theta$.

Demonstration.—Putting $\frac{1}{n}$ for n in De Moivre's formula we have

$$(\cos\theta + \sqrt{-1}\,\sin\theta)^{\frac{1}{n}} = \cos\tfrac{1}{n}\,\theta + \sqrt{-1}\,\sin\tfrac{1}{n}\,\theta.$$

Hence $\qquad \mathrm{OP}^{\frac{1}{n}} = \cos\frac{1}{n}\,\theta + \sqrt{-1}\,\sin\frac{1}{n}\,\theta,$

or $\mathrm{OP}^{\frac{1}{n}} = \mathrm{O}R$ revolved through $\frac{1}{n}\,\theta$.

Revolving OR through $\frac{1}{n}$ of θ gives but one of the n^{th} roots of OP, which I shall call the first of the n roots. The following proposition shows how to get any one of the remaining $n - 1$ roots. Let $m > 1$ and $< n + 1$.

Proposition 3.—The following equations indicate the $n - 1$ of the n roots of OP after the first:

The second of the n roots of $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ revolved through $\frac{1}{n}$ C,

“ third “ “ n “ “ $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ “ “ $\frac{2}{n}$ C,

“ fourth “ “ n “ “ $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ “ “ $\frac{3}{n}$ C,

“ m^{th} “ “ n “ “ $\mathrm{OP} = \mathrm{OP}^{\frac{1}{n}}$ “ “ $\frac{m-1}{n}$ C.

We can make the reference to OR by adding $\frac{1}{n}\,\theta$.

The m^{th} of the n roots of OP $=$ OR revolved through $\frac{1}{n}\,\theta + \frac{1}{n}(m - 1)\mathrm{C}$.

Demonstration.—Put OP_m to represent the m^{th} of the n roots of OP, then if the proposition is true $OP^n_m = OP$.

By prop. 1 $OP^n_m = OR$ revolved through $n(\frac{1}{n}\theta + \frac{1}{n}(m - 1)C$,

or $OP^n_m = OR$ " " θ,

since complete circumferences may be rejected.

But $OP = OR$ revolved through θ, therefore $OP^n_m = OP$.

Proposition 4.—$OR^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C.

Hence when $OP = OR^{\frac{1}{n}}$, OP is the n^{th} root of $+ 1$, or OR.

Demonstration.—When $\theta = RP$ becomes C, OP becomes OR, hence by substitution $OP^{\frac{1}{n}} = OR$ revolved through $\frac{1}{n}$ C (prop. 2),

becomes $OR^{\frac{1}{n}} = OR$ " " $\frac{1}{n}$ C.

This formula gives but one of the n roots of $+ 1$, or OR, which root I call the first of the n roots of $+ 1$.

By prop, 3 the m^{th} of the $n - 1$ remaining roots of $+ 1$, is the m^{th} power of the n^{th} root of $+ 1$.

It is plain that the n roots of $+ 1$ are *numerically* equal to each other but *geometrically* or *positionally* different.

The reader may test the above propositions by solving the binomial equations $x^2 = 1$, $x^3 = 1$, $x^4 = 1$, $x^5 = 1$, $x^6 = 1$, $x^8 = 1$, $x^{10} = 1$, $x^{12} = 1$, $x^{20} = 1$, etc., and constructing the roots in accordance with the convention that $\sqrt{-1}$ denotes *perpendicularity*. See my first article in the January Analyst The passage from $OR = 1$ to $OR = a$ is simple. The next article will deal with imaginary exponents.

* * *

SOLUTION OF AN INDETERMINATE PROBLEM.

BY GEO. R. PERKINS, L L. D., UTICA, N. Y,

No. 5.—*Find three right triangles having equal perimeters, and whose areas shall be in arithmetical progression.*

Since these triangles have the same perimeters, and their areas are in arithmetical progression, it is evident that the radii of their inscribed circles must be in arithmetical progression. We will therefore assume as follows: